Approximating Partition in Near-Linear Time

Jiayi Lian
(joint work with Lin Chen, Yuchen Mao, and Guochuan Zhang)

Zhejiang University

March 26, 2024

Problems

- Subset Sum
  - Given a multi-set X of n positive integers and a target t,
  - asks for a subset $Y \subseteq X$ with maximum $\Sigma(Y)$ that does not exceed t.
- Partition
  - $t = \Sigma(X)/2$

$$\Sigma(Y) = \sum_{y \in Y} y$$

- Both are NP-hard.

Approximation Scheme

- Standard $(1-\varepsilon)$-Approximation for Subset Sum

$$(1-\varepsilon)\Sigma(Y^*) \leqslant \Sigma(Y) \leqslant t.$$

  - [Kellerer et al. '97] $\widetilde{O}(n + 1/\varepsilon^2)$
  - [Bringmann & Nakos '21] proved it is the best possible assuming the (min, +)-convolution conjecture.

Approximation Scheme

- Standard $(1-\varepsilon)$-Approximation for Subset Sum

$$(1-\varepsilon)\Sigma(Y^*) \leqslant \Sigma(Y) \leqslant t.$$

  - [Kellerer et al. '97] $\widetilde{O}(n + 1/\varepsilon^2)$
  - [Bringmann & Nakos '21] proved it is the best possible assuming the (min, +)-convolution conjecture.

- $(1-\varepsilon)$-Approximation for Partition
  - [Deng et al. '23 & Wu and Chen'22] $\widetilde{O}(n + 1/\varepsilon^{5/4})$
  - [Abboud et al. '22] Conditional LB: $\mathrm{poly}(n)/\varepsilon^{1-o(1)}$.

Current Results

Table 1: Approximation schemes for Partition

Running Time		Reference
$\widetilde{O}(n + 1/\varepsilon^2)$	Deterministic	[Gens & Levner '80]
$\widetilde{O}(n + 1/\varepsilon^{5/3})$	Randomized	[Mucha, Węgrzycki & Włodarczyk '19]
$\widetilde{O}(n + 1/\varepsilon^{3/2})$	Deterministic	[Bringmann & Nakos '21]
$\widetilde{O}(n + 1/\varepsilon^{5/4})$	Deterministic	[Deng, Jin & Mao '23][Wu & Chen '22]
Conditional Lower Bound: $\mathrm{poly}(n)/\varepsilon^{1-o(1)}$ [Abboud et al. '22]		

Current Results

Table 1: Approximation schemes for Partition

Running Time		Reference
$\widetilde{O}(n + 1/\varepsilon^2)$	Deterministic	[Gens & Levner '80]
$\widetilde{O}(n + 1/\varepsilon^{5/3})$	Randomized	[Mucha, Węgrzycki & Włodarczyk '19]
$\widetilde{O}(n + 1/\varepsilon^{3/2})$	Deterministic	[Bringmann & Nakos '21]
$\widetilde{O}(n + 1/\varepsilon^{5/4})$	Deterministic	[Deng, Jin & Mao '23][Wu & Chen '22]
$\widetilde{O}(n + 1/\varepsilon)$	Randomized	this talk
Conditional Lower Bound: $\mathrm{poly}(n)/\varepsilon^{1-o(1)}$ [Abboud et al. '22]		

Basic Idea

- Compute $\mathcal{S}_X := \{\Sigma(Y) : Y \subseteq X\}$

$$\boxed{\Sigma(Y) = \Sigma_{y \in Y} y}$$

- Optimaze: find $s \in \mathcal{S}_X$ such that s is maximum but $s \leqslant t$.

Basic Idea

- Compute $\mathcal{S}_X := \{\Sigma(Y) : Y \subseteq X\}$

$\boxed{\Sigma(Y) = \Sigma_{y \in Y} y}$

- Optimaze: find $s \in \mathcal{S}_X$ such that s is maximum but $s \leqslant t$.

Sumset: $A + B = \{a + b : a \in A \cup \{0\}, b \in B \cup \{0\}\}$.

Let $X = \{x_1, ..., x_n\}$, $\mathcal{S}_X = \{x_1\} + ... + \{x_n\}$.

Basic Idea

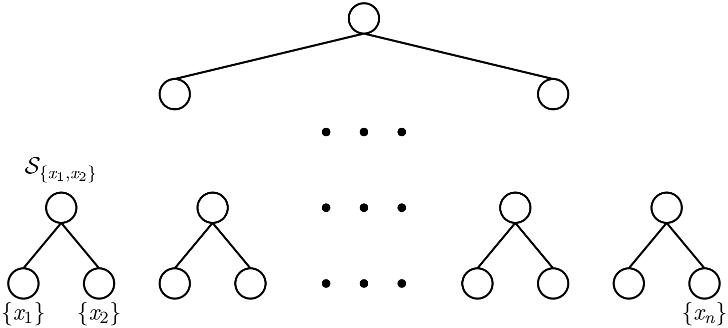
- Compute $\mathcal{S}_X := \{\Sigma(Y) : Y \subseteq X\}$

$\Sigma(Y) = \Sigma_{y \in Y} y$

- Optimaze: find $s \in \mathcal{S}_X$ such that s is maximum but $s \leqslant t$.

Sumset: $A + B = \{a + b : a \in A \cup \{0\}, b \in B \cup \{0\}\}$.

Let $X = \{x_1, ..., x_n\}$, $\mathcal{S}_X = \{x_1\} + ... + \{x_n\}$.

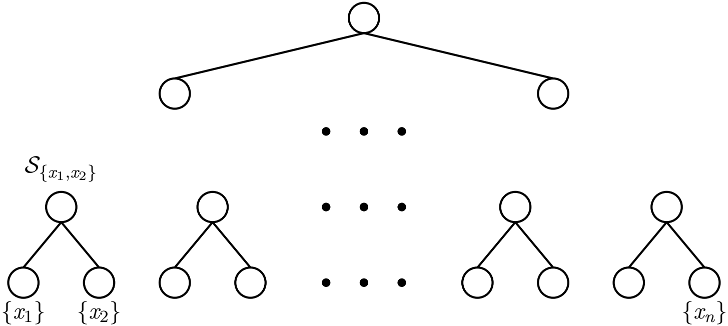


Basic Idea

$\mathcal{S}_X = \{x_1\} + ... + \{x_n\}$.

Sparse convolution: compute $A + B$ in $\widetilde{O}(|A + B|)$ time.

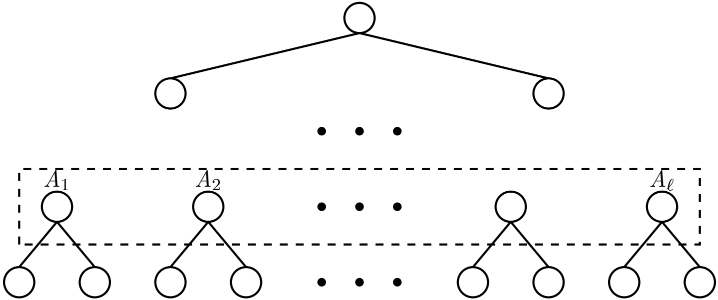
$|\mathcal{S}_X|$ can be $O(\Sigma x_i) = O(nw)$ or $O(t)$ by cutting. ($w = \max x_i$)



Basic Idea

Compute $\mathcal{S}_X := \{\Sigma(Y) : Y \subseteq X\}$. For each level:

- if $\Sigma|A_i| \leqslant \widetilde{\Theta}(u)$, compute it by sparse convolution;
- else, we show t is approximately in $\mathcal{S}_X$ by additive combinatoric.



Main Questions

- How to make $u = O(\frac{1}{\varepsilon})$?

  So sparse convolution can be done in $\widetilde{O}(n + \frac{1}{\varepsilon})$.

- How to use additive combinatoric?

  Then we can say t is approximately in $\mathcal{S}_X$.

Main Questions

- How to make $u = O(\frac{1}{\varepsilon})$?

  So sparse convolution can be done in $\widetilde{O}(n + \frac{1}{\varepsilon})$.

  - make $\max x_i = O(\frac{1}{\varepsilon})$,
  - scale and round during the tree.

- How to use additive combinatoric?

  Then we can say t is approximately in $\mathcal{S}_X$.

Make $\max x_i = O(1/\varepsilon)$

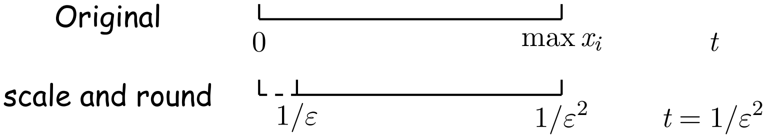
Scaling, Rounding, Grouping

Key point: for each element, we can round ε fraction of it.

Make $\max x_i = O(1/\varepsilon)$

Scaling, Rounding, Grouping

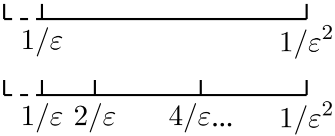
Key point: for each element, we can round ε fraction of it.



Make $\max x_i = O(1/\varepsilon)$

Scaling, Rounding, Grouping

Key point: for each element, we can round ε fraction of it.



$t = 1/\varepsilon^2$

group

$t = 1/\varepsilon^2$

$$X_\alpha = X \cap [\frac{\alpha}{\varepsilon}, \frac{2\alpha}{\varepsilon}]. \qquad X = X_1 \cup ... \cup X_{\log(1/\varepsilon)}.$$

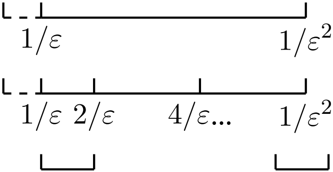
$$\mathcal{S}_X = \mathcal{S}_{X_1} + ... + \mathcal{S}_{X_{\log(1/\varepsilon)}}.$$

This can be done approximately in $O(1/\varepsilon)$.

Make $\max x_i = O(1/\varepsilon)$

Scaling, Rounding, Grouping

Key point: for each element, we can round ε fraction of it.



$t = 1/\varepsilon^2$

group $t = 1/\varepsilon^2$

scale for each group $[\frac{\alpha}{\varepsilon}, \frac{2\alpha}{\varepsilon}]$ $[\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$

$$t = \frac{1}{\varepsilon^2} \quad \Rightarrow t' = \frac{t}{\alpha} \in [\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}]$$

Make $\max x_i = O(1/\varepsilon)$

We can reduce to this problem:

- $X \subseteq [1/\varepsilon, 2/\varepsilon]$,
- $t \in [1/\varepsilon, 1/\varepsilon^2]$,
- $t \leqslant \Sigma(X)/2$ (Otherwise, we just let $t = \Sigma(X)/2$),
- BUT now, we have to compute $\mathcal{S}_X[0, t]$
  - If it's always sparse, we actually compute $\mathcal{S}_X[0, t]$,
  - NOW for the dense part:

Additive Combinatorics

[Szemerédi & Vu '05]

There exists a constant c such that the following holds. Let $A_1, ..., A_\ell$ be subsets of $[1, u]$ of size at least $|A|$. If $\ell|A| \geqslant cu$, then $A_1 + \cdots + A_\ell$ contains an arithmetic progression of length at least u.

Additive Combinatorics

[Szemerédi & Vu '05]

There exists a constant c such that the following holds. Let $A_1, ..., A_\ell$ be subsets of $[1, u]$ of size at least $|A|$. If $\ell|A| \geqslant cu$, then $A_1 + \cdots + A_\ell$ contains an arithmetic progression of length at least u.

Corollary

There exists a constant c such that: Let $A_1, ..., A_\ell$ be subsets of $[1, u]$. If $\Sigma_{i=1}^{\ell}|A_i| \geqslant cu \log u$, then $A_1 + \cdots + A_\ell$ contains an arithmetic progression of length at least u.

- Common difference $\Delta \leqslant \dfrac{\ell \cdot u}{u} \leqslant \ell$.

Using Additive Combinatorics

What we want to do?

If $s_1 \leqslant t \leqslant s_n$ and $s_i - s_{i-1} \leqslant O(\varepsilon t)$,

We can say that $t \in \mathcal{S}_X$ approximately.

Using Additive Combinatorics

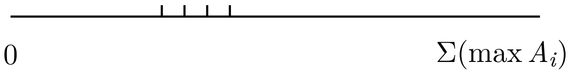
What we want to do?

If $s_1 \leqslant t \leqslant s_n$ and $s_i - s_{i-1} \leqslant O(\varepsilon t)$,

We can say that $t \in \mathcal{S}_X$ approximately.

However, If $\Sigma_{i=1}^{\ell} |A_i| \geqslant cu \log u$.

Just dense

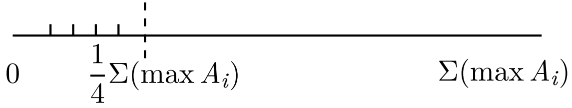


$$s_1 =? \quad s_n =? \quad s_i - s_{i-1} = \Delta = O(\ell), \quad s_n - s_i \geqslant u$$

Using Additive Combinatorics

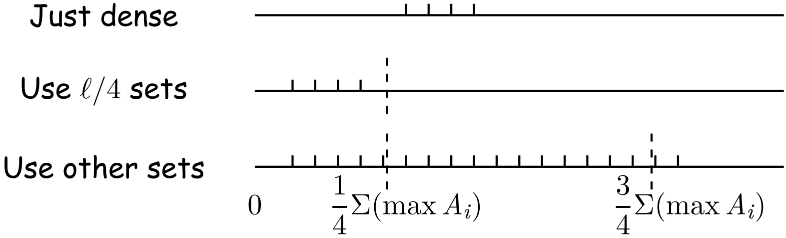
If $\Sigma_{i=1}^{\ell}|A_i| \geqslant 4cu\log u$, we can choose at most $\frac{\ell}{4}$ sets that have an arithmetic progression.
(We choose ones with the smallest maximum element)

Use $\ell/4$ sets



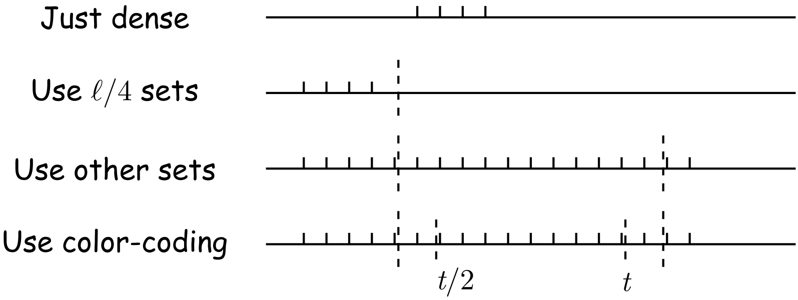
$$s_1 \leqslant s_n = \frac{1}{4}\Sigma(\max A_i), \quad s_i - s_{i-1} = \Delta = O(\ell), \quad s_n - s_i \geqslant u$$

Using Additive Combinatorics



$$s_1 \leqslant \frac{1}{4}\Sigma(\max A_i), \quad s_n \geqslant \frac{3}{4}\Sigma(\max A_i), \quad s_i - s_{i-1} \leqslant \Delta = O(\ell)$$

Using Additive Combinatorics



By color-coding, we can let $t = \Theta(\frac{1}{2}\Sigma(\max A_i))$ and $l = O(\varepsilon t)$.

$$s_1 \leqslant \frac{t}{2}, \qquad s_n \geqslant t, \qquad s_i - s_{i-1} \leqslant O(\varepsilon t)$$

Approximating $\mathcal{S}[t/2, t]$

If $A_1 + ... + A_\ell$ has a sequence $s_1, ..., s_n$ that:

$$s_1 \leqslant \frac{t}{2}, \qquad s_n \geqslant t, \qquad s_i - s_{i-1} \leqslant O(\varepsilon t).$$

Then

$$\widetilde{\mathcal{S}} = \{t/2 + \varepsilon t i : t \in [0, \frac{1}{2\varepsilon}]\}$$

approximates $\mathcal{S}[t/2, t]$ with additive error εt.

Approximating $\mathcal{S}[t/2, t]$

If $A_1 + ... + A_\ell$ has a sequence $s_1, ..., s_n$ that:

$$s_1 \leqslant \frac{t}{2}, \qquad s_n \geqslant t, \qquad s_i - s_{i-1} \leqslant O(\varepsilon t).$$

Then

$$\widetilde{\mathcal{S}} = \{t/2 + \varepsilon t i : t \in [0, \frac{1}{2\varepsilon}]\}$$

approximates $\mathcal{S}[t/2, t]$ with additive error εt.

For $\mathcal{S}[1, t/2]$, again, we use grouping:

Compute $\mathcal{S}[1, 2], \mathcal{S}[2, 4], \mathcal{S}[4, 8]$... and merge. ($\log(1/\varepsilon)$ times)

Color Coding

- Let $Y \subseteq X$ that $\Sigma(Y) = t$. We have $X \in [1/\varepsilon, 2/\varepsilon]$.
- $|Y| \leqslant \varepsilon t$. Let $|Y| = k$.
- If $t \ll \Sigma(X)$, Y just uses a few elements in X.

Color Coding

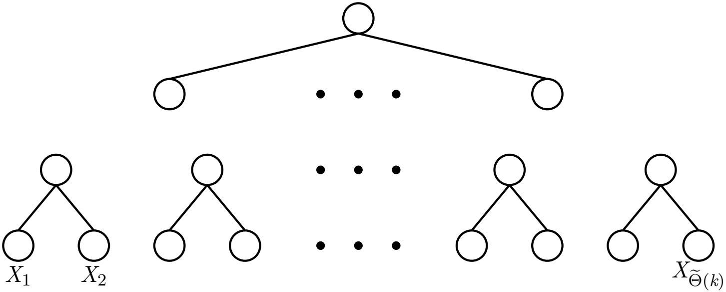
- Let $Y \subseteq X$ that $\Sigma(Y) = t$. We have $X \in [1/\varepsilon, 2/\varepsilon]$.
- $|Y| \leqslant \varepsilon t$. Let $|Y| = k$.
- If $t \ll \Sigma(X)$, Y just uses a few elements in X.
- Color Coding: [Bringmann '17]
  - Randomly partition X to $\widetilde{\Theta}(k)$ subset.
  - With high probability, $|Y \cap X_i| \leqslant 1$ for all i.

Color Coding

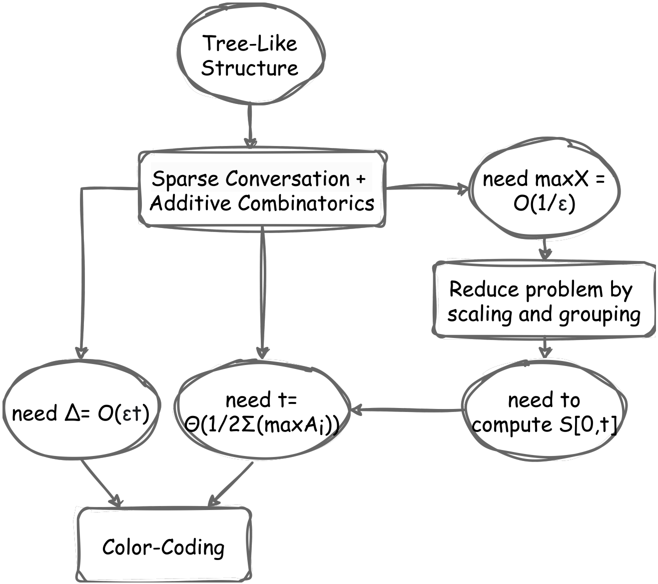
- Let $Y \subseteq X$ that $\Sigma(Y) = t$. We have $X \in [1/\varepsilon, 2/\varepsilon]$.
- $|Y| \leqslant \varepsilon t$. Let $|Y| = k$.
- If $t \ll \Sigma(X)$, Y just uses a few elements in X.
- Color Coding: [Bringmann '17]
  - Randomly partition X to $\widetilde{\Theta}(k)$ subset.
  - With high probability, $|Y \cap X_i| \leqslant 1$ for all i.
  - With high probability, $\Sigma(Y) \in X_1 + \ldots + X_{\widetilde{\Theta}(k)}$.

Color Coding

- With high probability, $\Sigma(Y) \in X_1 + \ldots + X_{\widetilde{\Theta}(k)}$.
- For any $s \in \mathcal{S}_X$, with high probability, $s \in X_1 + \ldots + X_{\widetilde{\Theta}(k)}$.
- $\Sigma(\max X_i) = \widetilde{\Theta}(k \cdot \frac{1}{\varepsilon}) = \widetilde{\Theta}(t)$.
- Common distance $l = O(\varepsilon t)$



Technical Overview



Open Questions

- Does Partition admit a deterministic approximation scheme in time $\widetilde{O}(n + 1/\varepsilon)$?
- Does Subset Sum admit a deterministic weak approximation scheme in time $\widetilde{O}(n + 1/\varepsilon)$?
- Does Subset Sum (or Partition) have an exact algorithm in time $\widetilde{O}(n + w)$? (w is the maximum element in X.)

Reference

[1] Amir Abboud et al. "SETH-based Lower Bounds for Subset Sum and Bicriteria Path". In: ACM Transactions on Algorithms 18.1 (Jan. 2022), pp. 1–22. ISSN: 1549-6325, 1549-6333.

[2] Karl Bringmann. "A Near-Linear Pseudopolynomial Time Algorithm for Subset Sum". In: SODA 2017. Jan. 2017, pp. 1073–1084.

[3] Karl Bringmann, Nick Fischer, and Vasileios Nakos. "Deterministic and Las Vegas Algorithms for Sparse Nonnegative Convolution". In: SODA 2022. Proceedings. Jan. 2022, pp. 3069–3090.

[4] Karl Bringmann and Vasileios Nakos. "A Fine-Grained Perspective on Approximating Subset Sum and Partition". In: SODA 2021. Proceedings. Jan. 2021, pp. 1797–1815.

[5] Karl Bringmann and Vasileios Nakos. "Fast N-Fold Boolean Convolution via Additive Combinatorics". In: ICALP 2021. 2021.

[6] Mingyang Deng, Ce Jin, and Xiao Mao. "Approximating Knapsack and Partition via Dense Subset Sums". In: SODA 2023. Proceedings. Jan. 2023, pp. 2961–2979.

[7] Marcin Mucha, Karol Węgrzycki, and Michał Włodarczyk. "A Subquadratic Approximation Scheme for Partition". In: SODA 2019. Proceedings. Jan. 2019, pp. 70–88.

[8] E. Szemerédi and V. Vu. "Long Arithmetic Progressions in Sumsets: Thresholds and Bounds". In: Journal of the American Mathematical Society 19.1 (Sept. 2005), pp. 119–169. ISSN: 0894-0347, 1088-6834.

[9] Xiaoyu Wu and Lin Chen. "Improved Approximation Schemes for (Un-)Bounded Subset-Sum and Partition". In: arXiv:2212.02883. Dec. 2022. arXiv: 2212.02883 [cs].

Thank you!