A Nearly Quadratic-Time FPTAS for Knapsack

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Knapsack

- n items with weights $\{w_i\}_i$ and profits $\{p_i\}_i$
- a knapsack with capacity t
- maximize total profit subject to the capacity constraint

$$\max\left\{\sum_{i=1}^{n} p_i x_i : \sum_{i=1}^{n} w_i x_i \leqslant t,\, x_i \in \{0,1\}\right\}$$

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- NP-hard

Approximation Scheme

- FPTAS: fully polynomial-time approximation scheme
- for any instance I and any $\varepsilon > 0$,

$$\mathrm{ALG}(I, \varepsilon) \geqslant (1 - \varepsilon)\mathrm{OPT}(I)$$

- runs in $\mathrm{poly}(|I|, 1/\varepsilon)$ time

Current work

$O(n \log n + (\frac{1}{\varepsilon})^4 \log \frac{1}{\varepsilon})$	[Ibarra & Kim '75]
$O(n \log n + (\frac{1}{\varepsilon})^4)$	[Lawler '79]
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^3 \log^2 \frac{1}{\varepsilon})$	[Kellerer & Pferschy '04]
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{5/2} \log^3 \frac{1}{\varepsilon})$	[Rhee '15]
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{12/5} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	[Chan '18]
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	[Jin '19]
$\widetilde{O}(n + (\frac{1}{\varepsilon})^{11/5} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	[Deng, Jin & Mao '23]

Conditional lower bound $\Omega((n + 1/\varepsilon)^{2-\delta})$ for any $\delta > 0$

[Künnemann, Paturi & Schneider '17]

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$\widetilde{O}(n + (\frac{1}{\varepsilon})^2)$	Our Work & [Mao '24]

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Value Function

- Let I be a set of items.
- Define $f_I : \{0, 1, 2, \ldots, t\} \to \mathbb{Z}$ as follows.

$$f_I(y) = \max \left\{ \sum_{i \in I} p_i x_i : \sum_{i \in I} w_i x_i \leqslant y, x_i \in \{0, 1\} \right\}$$

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- We want to compute $f_I(t)$ approximately.

Compute Value Function

- Let I_1 and I_2 be a partition of I.
- We have for any $y \in \{0, \ldots, t\}$

$$f_I(y) = \max\{f_{I_1}(y_1) + f_{I_2}(y_2) : y_1 + y_2 = y\}$$

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- $(\max, +)$-convolution

$$f_I = f_{I_1} \oplus f_{I_2}.$$

- Approximate the whole function f_{I_1} and f_{I_2} rather than a single value $f_{I_1}(t)$ or $f_{I_2}(t)$.

Approximate a function

- $\widetilde{f}$ approximate f with factor $1+\varepsilon$ if for any $y \in \{0, \dots, t\}$,

$$0 \leqslant f(y) - \widetilde{f}(y) \leqslant \varepsilon \cdot \widetilde{f}(y).$$

- $\widetilde{f}$ approximate f with additive error δ if for any $y \in \{0, \dots, t\}$,

$$0 \leqslant f(y) - \widetilde{f}(y) \leqslant \delta.$$

Main Idea

- If $f_I(t) \geqslant B$,
  - compute a $\widetilde{f}$ approximate f with additive error εB.
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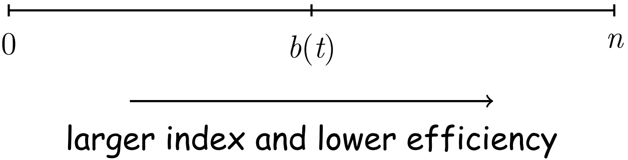
  - equal to approximate f_{I_1} with factor $1 + \Theta(\varepsilon \frac{B}{B_1})$.
- allow a Large factor !

An Additive Combinatorics Result

- Two multi-sets A and B of integers $[1, p_{\max}]$.
- If $|\text{supp}(A)| \geqslant \widetilde{\Omega}(p_{\max}^{1/2})$ and $\Sigma(B) \geqslant \widetilde{\Omega}(p_{\max}^{3/2})$, then $\exists A' \subseteq A, B' \subseteq B$ such that $\Sigma(A') = \Sigma(B')$.

[Chen, Lian, Mao & Zhang '24]

A Proximity Result



- break item $b(t)$ for capacity t

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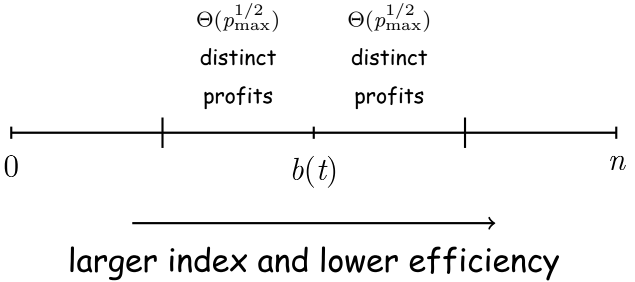

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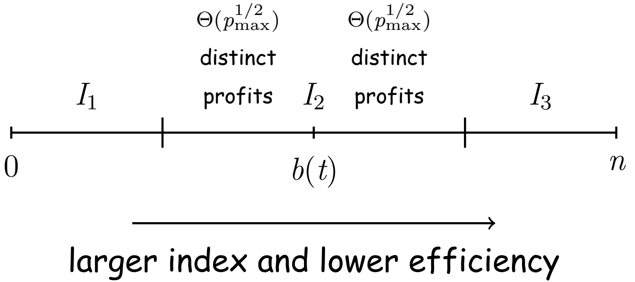
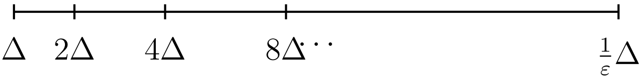


- break item $b(t)$ for capacity t
- By additive combinatorics, $p(I_1^-) \leqslant O(p_{\max}^{3/2})$ and $p(I_3^+) \leqslant O(p_{\max}^{3/2})$
  - $|\text{supp}(I_2)| = \Theta(p_{\max}^{1/2})$ and if $p(I_1^-), p(I_3^+) \geqslant \widetilde{\Omega}(p_{\max}^{3/2})$

Bound the maximum profit

- Partition the items into $I_1, I_2, \ldots$ by their profit

$$I_j = \left\{ i \in I : p_i \in \left(2^{j-1} \cdot \Delta, 2^j \cdot \Delta\right] \right\}, \quad \Delta = \Theta(\varepsilon \cdot \text{opt}).$$



$$f_I = f_{I_1} \oplus f_{I_2} \oplus \cdots \oplus f_{I_{\log 1/\varepsilon}}$$

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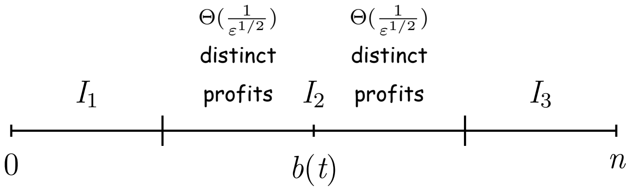
- for each j, scale the profits to $(\frac{1}{\varepsilon}, \frac{2}{\varepsilon}]$.
- Round to integers.

A Reduced Instance

- $p_i \in \left(\frac{1}{\varepsilon}, \frac{2}{\varepsilon}\right] \cap \mathbb{Z}$
- $f_I(t) \in [\frac{1}{\varepsilon^2} \frac{2}{\varepsilon^2}]$
- GOAL: approximate f_I with factor $\widetilde{O}(\varepsilon)$, or with absolute error

$$\widetilde{O}(\varepsilon) \cdot f_I(t) = \widetilde{O}(\frac{1}{\varepsilon})$$

The Proximity Result



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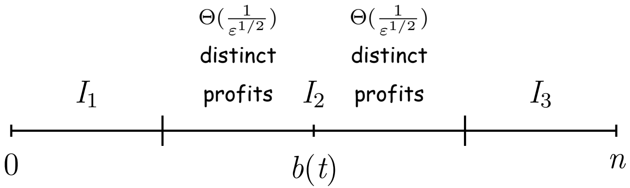

- $p(I_1^-) \leqslant O(\frac{1}{\varepsilon^{3/2}})$ and $p(I_3^+) \leqslant O(\frac{1}{\varepsilon^{3/2}})$
- Approximate $f_{I_1}, f_{I_2}, f_{I_3}$ and $f_I = f_{I_1} \oplus f_{I_2} \oplus f_{I_3}$.

Approximate f_{I_2}:

- there are only $\Theta(\frac{1}{\varepsilon^{1/2}})$ distinct profits.
- can be computed in $\widetilde{O}(\frac{1}{\varepsilon}m^2) = \widetilde{O}(\frac{1}{\varepsilon^2})$ time where $m = \Theta(\frac{1}{\varepsilon^{1/2}})$

[Chan' 18]
$f_{I_1} \oplus f_{I_2} \oplus \cdots \oplus f_{I_m}$ can be $(1+\varepsilon)$ approximated in $\widetilde{O}(\frac{1}{\varepsilon}m^2)$ time if the items in each I_i have the same profit.

Approximate f_{I_3} and f_{I_1}:

- it suffices to approximate $\min(f_{I_3}, O(\frac{1}{\varepsilon^{3/2}}))$
- allow a "large" approximation factor.
  - the absolute error allowed is $\widetilde{O}(\varepsilon) \cdot f_I(t) = \widetilde{O}(\frac{1}{\varepsilon})$.
  - the approximation factor now is $1 + \widetilde{O}(\varepsilon^{1/2})$.
  - allow rescaling and rounding: $p'_i \in (\frac{1}{\varepsilon^{1/2}}, \frac{2}{\varepsilon^{1/2}}] \cap \mathbb{Z}$.

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  - allow rescaling and rounding: $p'_i \in (\frac{1}{\varepsilon^{1/2}}, \frac{2}{\varepsilon^{1/2}}] \cap \mathbb{Z}$.
- computed by standard dynamic programming in $\widetilde{O}(\frac{1}{\varepsilon^2})$ time.

End of Story?

We need to approximate f_I on all capacities.

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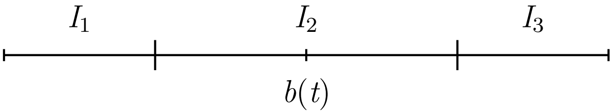
But, proximity result only works for a single capacity.



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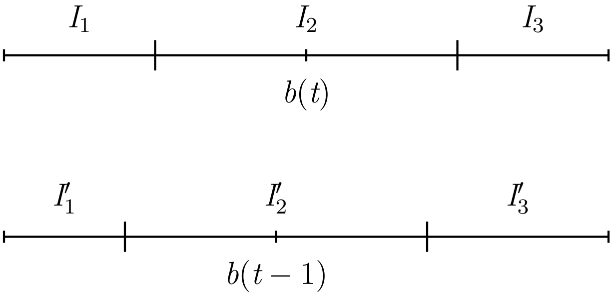

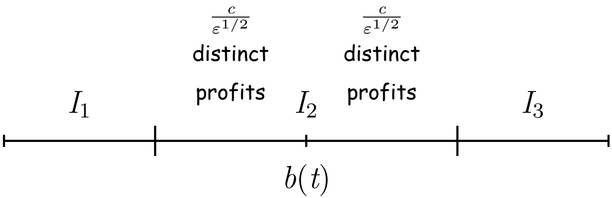
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I_1
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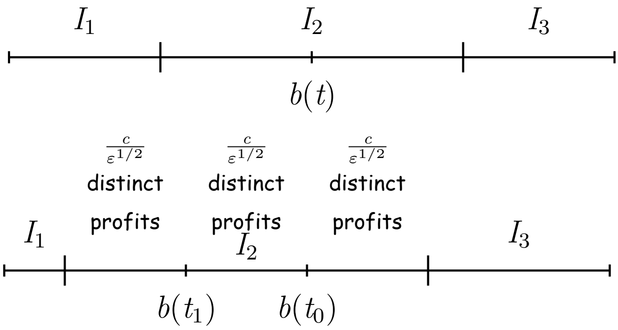
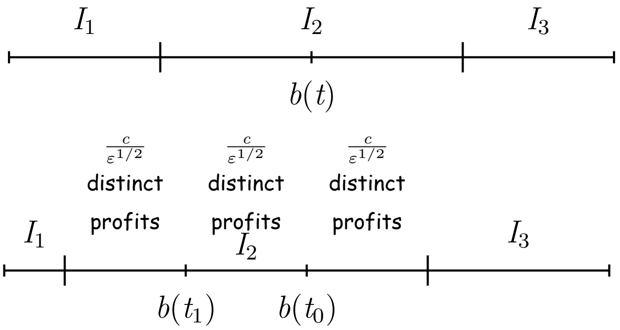
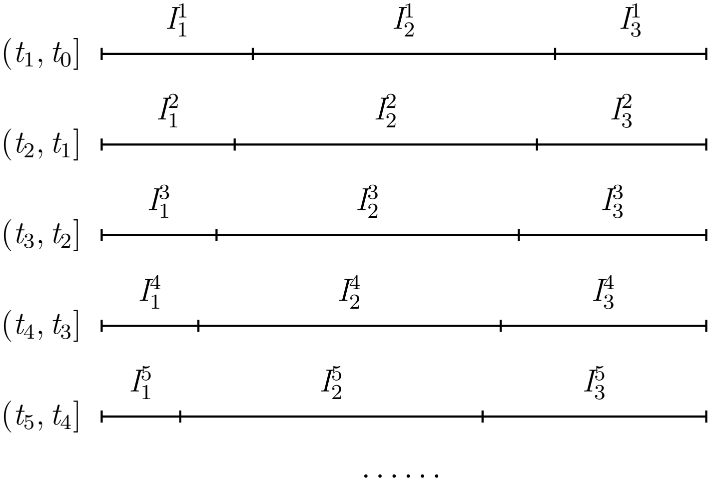


- All $t' \in [t_1, t_0]$ share the same partition (I_1, I_2, I_3).



- All $t' \in [t_1, t_0]$ share the same partition (I_1, I_2, I_3).
- partition $[0, t]$ into $O(\frac{1}{\varepsilon^{1/2}})$ intervals.



I_1^1 I_2^1 I_3^1
$(t_1, t_0]$
I_1^2 I_2^2 I_3^2
$(t_2, t_1]$
I_1^3 I_2^3 I_3^3
$(t_3, t_2]$
I_1^4 I_2^4 I_3^4
$(t_4, t_3]$
I_1^5 I_2^5 I_3^5
$(t_5, t_4]$
$\cdots\cdots$

- Compute I_1^j and I_3^j
  - for $j \in [1, \theta]$, $I_1^j \subseteq I_1^{j-1}$, $I_3^{j-1} \subseteq I_3^j$
  - Can be computed in $\widetilde{O}(\frac{1}{\varepsilon^2})$ by dynamic programming

- Compute I_2^j
  - $f_{I_2^j} \leqslant |I_2^j| \cdot \frac{2}{\varepsilon}$.
  - the approximation factor can be $1 + \frac{1}{|I_2^j|}$.

Recall [Chan' 18]
$f_{I_1} \oplus f_{I_2} \oplus \cdots \oplus f_{I_m}$ can be $(1+\varepsilon)$ approximated in $\widetilde{O}(\frac{1}{\varepsilon} m^2)$ time if the items in each I_i have the same profit.

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  - $\sum_j |I_2^j| \leqslant 2b(t) = O(\frac{1}{\varepsilon})$.

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Summary

Key: small contribution = large approximation factor

- Use additive combinatorics results.
  - Reduce Problem such that $p_{\max} = \Theta(\frac{1}{\varepsilon})$.
  - $I = I_1 \cup I_2 \cup I_3$
  - we can compute them in $\widetilde{O}(\frac{1}{\varepsilon^2})$ time
  - proximity result only works for a single capacity.

Summary

Key: small contribution = large approximation factor

- Use additive combinatorics results.
- partition $[0, t]$ into intervals.
  - compute f_{I_1}, f_{I_3} for all intervals at the same time..
  - rescale f_{I_2} and compute all f_{I_2} in quadratic time.

Summary

Key: small contribution = large approximation factor

- Use additive combinatorics results.
- partition $[0, t]$ into intervals.
- Get a $\widetilde{O}(n + \frac{1}{\varepsilon^2})$ time FPTAS !

Open Problems

- Is there an FPTAS running in $O(n/\varepsilon)$ time?
  - $O((\frac{1}{\varepsilon})^2 n \log \frac{1}{\varepsilon})$ [Kellerer & Pferschy '99]
  - $\widetilde{O}(\frac{1}{\varepsilon} n^{3/2})$ [Chan '18]
- Is there an $O(n w_{\max})$-time algorithm?
- Is there an $O(n + (w_{\max} + p_{\max})^{2-\delta})$-time algorithm for some $\delta > 0$?

Thank You!