Weakly Approximating Knapsack in Subquadratic Time

Jiayi Lian

Joint work with Lin Chen, Yuchen Mao, and Guochuan Zhang

Zhejiang University

July 8, 2025

ICALP 2025

Knapsack

- n items with weights $\{w_i\}_i$ and profits $\{p_i\}_i$
- a knapsack with capacity t
- maximize total profit subject to the capacity constraint

$$\max \left\{ \sum_{i=1}^{n} p_i x_i : \sum_{i=1}^{n} w_i x_i \leq t, x_i \in \{0,1\} \right\}$$

PTAS

- Asks for a subset S of items with

$$\begin{aligned} p(S) &\geq OPT/(1+\varepsilon) \\ w(S) &\leq t \end{aligned}$$

- Can be done in $\tilde{O}(n + (\frac{1}{\varepsilon})^2)$ time
  [Chen, Lian, Mao & Zhang '24][Mao '24]

PTAS

- Asks for a subset S of items with

$$p(S) \geq OPT/(1+\varepsilon)$$
$$w(S) \leq t$$

- Can be done in $\tilde{O}(n + (\frac{1}{\varepsilon})^2)$ time
  [Chen, Lian, Mao & Zhang '24][Mao '24]

- No $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm for any constant $\delta > 0$, under the (min, +)-convolution conjecture
  [Künnemann, Paturi & Schneider '17]
  [Cygan, Mucha, Węgrzycki & Włodarczyk '19]

Resource Augmentation

- Asks for a subset S of items with

$$p(S) \geq OPT$$
$$w(S) \leq (1+\varepsilon)t$$

- Can be done in $\tilde{O}(n + (\frac{1}{\varepsilon})^2)$ time
- No $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm for any constant $\delta > 0$, under the (min, +)-convolution conjecture

Weak Approximation

- Asks for a subset S of items with

$$p(S) \geq OPT/(1+\varepsilon)$$
$$w(S) \leq (1+\varepsilon)t$$

- Can it done in $\tilde{O}(n + (\frac{1}{\varepsilon})^{2-\delta})$ time for some constant $\delta > 0$?

Related Problems

- Subset Sum:
  Standard approx: no $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n + 1/\varepsilon)$ time

Related Problems

- Subset Sum:
  Standard approx: no $O((n+\frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n+1/\varepsilon)$ time

- Unbounded Knapsack:
  Standard approx: no $O((n+\frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n+(\frac{1}{\varepsilon})^{3/2})$ time

Related Problems

- Subset Sum:
  Standard approx: no $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n + 1/\varepsilon)$ time
- Unbounded Knapsack:
  Standard approx: no $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n + (\frac{1}{\varepsilon})^{3/2})$ time
- Knapsack:
  Standard approx: no $O((n + \frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: ?

Related Problems

- Subset Sum:
  Standard approx: no $O((n+\frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n+1/\varepsilon)$ time

- Unbounded Knapsack:
  Standard approx: no $O((n+\frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: solved in $\tilde{O}(n+(\frac{1}{\varepsilon})^{3/2})$ time

- Knapsack:
  Standard approx: no $O((n+\frac{1}{\varepsilon})^{2-\delta})$-time algorithm
  Weak approx: ? solved in $\tilde{O}(n+(\frac{1}{\varepsilon})^{7/4})$ time

Value Function

- Let I be a set of items.
- Define $f_I : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows.

$$f_I(x) = \max \left\{ \sum_{i \in I} p_i z_i : \sum_{i \in I} w_i z_i \leq x,\, z_i \in \{0, 1\} \right\}$$

I

w	2	3
p	3	4

f

	0	1	2	3	4	5
	0	0	3	4	4	7

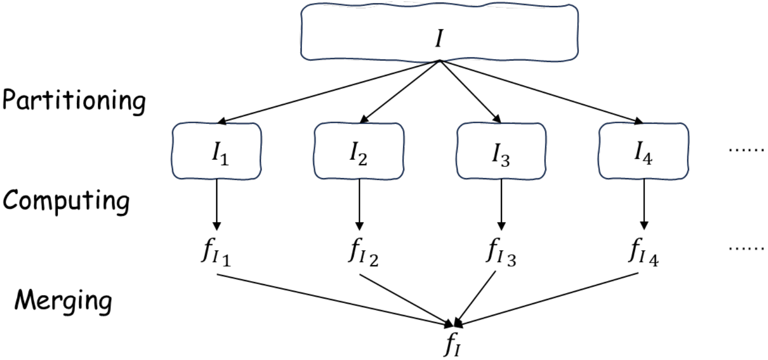
Value Function

- Let I be a set of items.
- Define $f_I : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ as follows.

$$f_I(x) = \max \left\{ \sum_{i \in I} p_i z_i : \sum_{i \in I} w_i z_i \leq x, z_i \in \{0,1\} \right\}$$

- Weak approx Knapsack $\Rightarrow$ Compute f' weak approx f_I:

Framework for Value Function

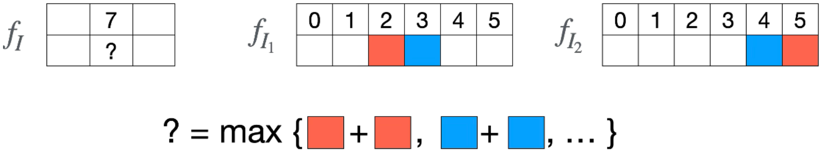


Max-Plus Convolution

- Given f_{I_1} and f_{I_2}

$$f_{I_1 \cup I_2}(x) = \max\left\{ f_{I_1}(x_1) + f_{I_2}(x_2) : x_1 + x_2 = x \right\}$$

- $f_{I_1 \cup I_2} = f_{I_1} \oplus f_{I_2}$: (max, +)-convolution



Max-Plus Convolution

- $f_{I_1 \cup I_2} = f_{I_1} \oplus f_{I_2}$: (max, +)-convolution
- General Case: can be computed in $O(n^2)$ time
  (min, +)-convolution conjecture: no $O(n^{2-\delta})$-time algorithm

Max-Plus Convolution

- $f_{I_1 \cup I_2} = f_{I_1} \oplus f_{I_2}$: (max, +)-convolution
- General Case: can be computed in $O(n^2)$ time
  (min, +)-convolution conjecture: no $O(n^{2-\delta})$-time algorithm
- Bounded Monotone: can be computed in $\widetilde{O}(n^{3/2})$ time
  [Chi, Duan, Xie & Zhang '22]

Max-Plus Convolution

- $f_{I_1 \cup I_2} = f_{I_1} \oplus f_{I_2}$: (max, +)-convolution
- General Case: can be computed in $O(n^2)$ time
  (min, +)-convolution conjecture: no $O(n^{2-\delta})$-time algorithm
- Bounded Monotone: can be computed in $\widetilde{O}(n^{3/2})$ time
  [Chi, Duan, Xie & Zhang '22]
  - Given monotone functions f_1 and f_2, a weak approximation of $f_1 \oplus f_2$ can be computed in $\tilde{O}((\frac{1}{\varepsilon})^{3/2})$ time
  - We can partition items into several groups

The Reduced Problem

- For some $\alpha \geq 1$,
  - $w_i \in [1, 2]$
  - $p_i \in [1, 2]$
  - $t = \Theta(\frac{1}{\alpha\varepsilon})$
  - absolute err $= O(\frac{1}{\alpha})$
- When $\alpha \geq 1/\varepsilon^{2/3}$, the solution size is small and it can be tackled using color coding
- The difficult case is when $1 \leq \alpha \leq 1/\varepsilon^{2/3}$
- Compute the value function of this set

Value Function with Bounded Efficiency

- item efficiency $= \frac{p_i}{w_i}$
- If item efficiencies are same,
  weak approx f_I = weak approx sumset sum —> in $\widetilde{O}(\frac{1}{\varepsilon})$ time

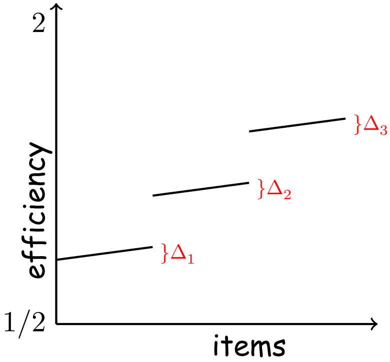
Value Function with Bounded Efficiency

- item efficiency $= \frac{p_i}{w_i}$
- If item efficiencies are same,
  weak approx f_I = weak approx sumset sum —> in $\widetilde{O}(\frac{1}{\varepsilon})$ time
- If item efficiencies are in $[\rho, \rho + \Delta]$, using 2D-FFT,
  f_I can be approximated in $\widetilde{O}((\frac{1}{\varepsilon})^2\Delta)$ time.

Value Function with Bounded Efficiency

- item efficiency $= \frac{p_i}{w_i}$
- If item efficiencies are same,
  weak approx f_I = weak approx sumset sum —> in $\widetilde{O}(\frac{1}{\varepsilon})$ time
- If item efficiencies are in $[\rho, \rho + \Delta]$, using 2D-FFT,
  f_I can be approximated in $\widetilde{O}((\frac{1}{\varepsilon})^2 \Delta)$ time.
  - Item efficiencies are in $[1/2, 2] \Rightarrow$ still quadratic time

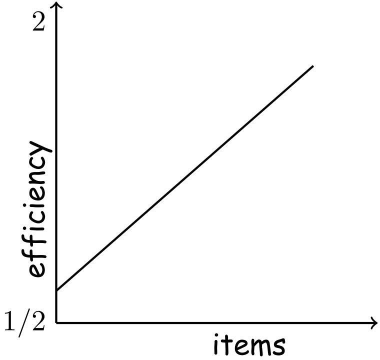
Items with Similar Efficiencies



- partition into groups
- compute their value function in time

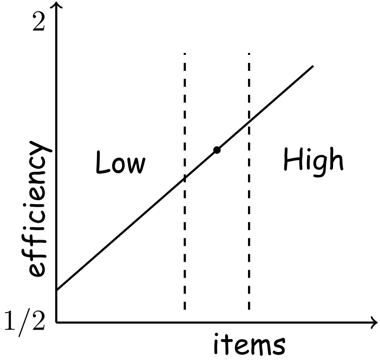
$$\tilde{O}((\frac{1}{\varepsilon})^2 \sum \Delta_i)$$

Items with Different Efficiencies



- use a proximity bound

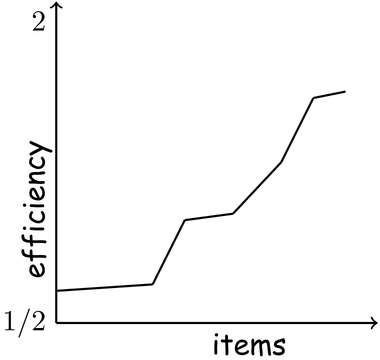
Items with Different Efficiencies



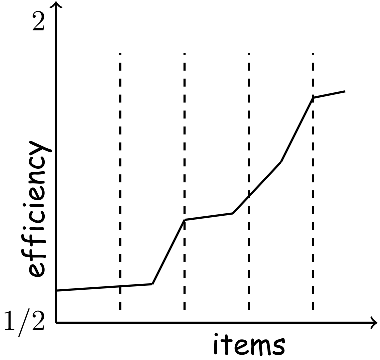
- use a proximity bound
- partition into three parts
- only a few items from Low part will contribute;

  and only a few items from High part will not contribute
- allow a larger approx factor

General Case

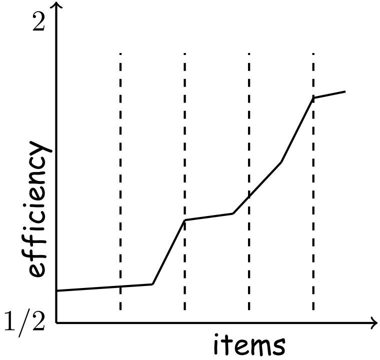


General Case



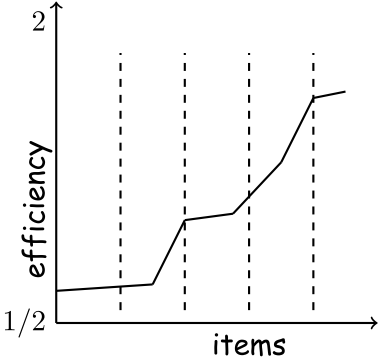
- partition into groups of fix size τ

General Case



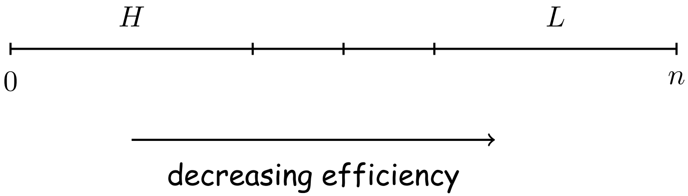
- partition into groups of fix size τ
- $\Delta \leq \frac{1}{\tau}$ (Good): Compute directly
- $\Delta \geq \frac{1}{\tau}$ (Bad): Use a larger factor

General Case



- partition into groups of fix size τ
- $\Delta \leq \frac{1}{\tau}$ (Good): Compute directly
- $\Delta \geq \frac{1}{\tau}$ (Bad): Use a larger factor
- In $\tilde{O}(n + (\frac{1}{\varepsilon})^{11/6})$ time

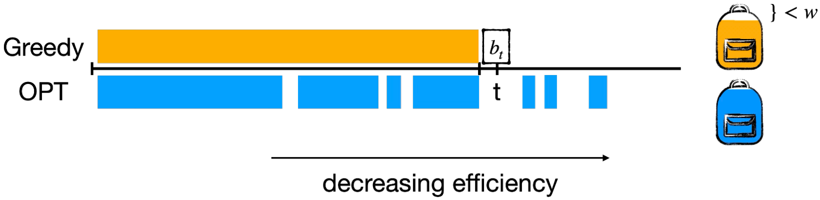
Proximity Bound



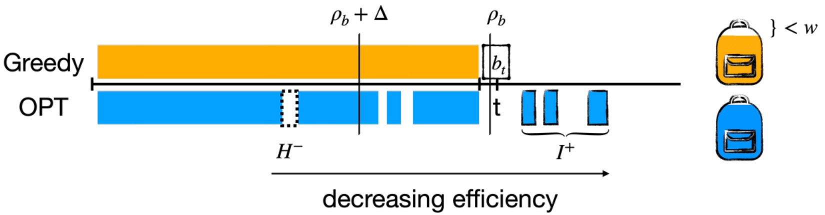
- Most items in H will be used
- Most items in L will not be used

Efficiency-Based Proximity

- Greedy Solution: add items until could not add
- Breaking item: the first item that the greedy solution does not used

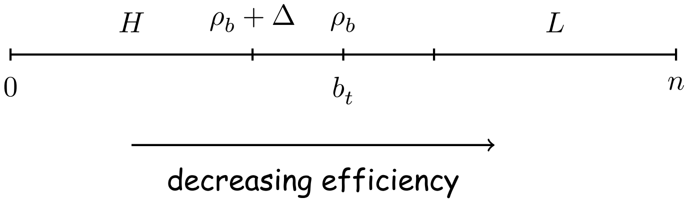


Efficiency-Based Proximity



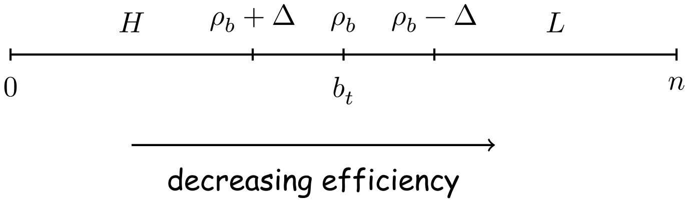
- Weight: $I^+ - H^- \leq w_b$
- Efficiency: $H^- - I^+ \geq \Delta$
- $w(H^-) \leq \frac{p_b}{\Delta}$

Efficiency-Based Proximity



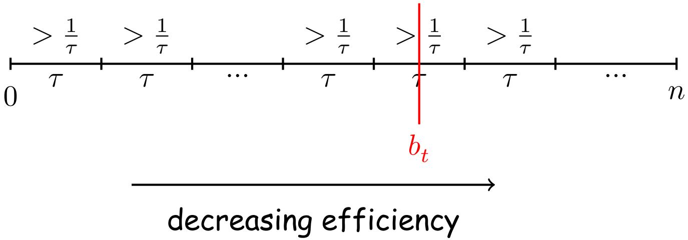
- $w(H^-) \leq \frac{p_{\max}}{\Delta} \leq \frac{2}{\Delta}$

Efficiency-Based Proximity

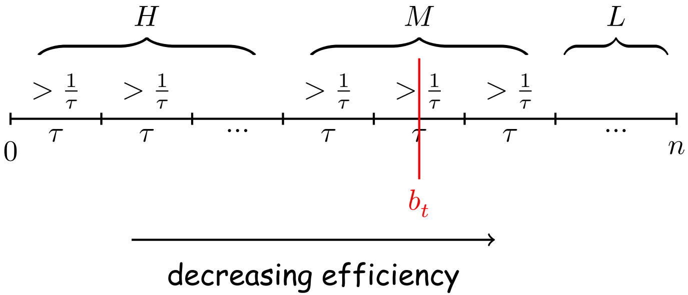


- $w(H^-) \leq \frac{p_{\max}}{\Delta} \leq \frac{2}{\Delta}$
- $w(L^+) \leq \frac{p_{\max}}{\Delta} \leq \frac{2}{\Delta}$

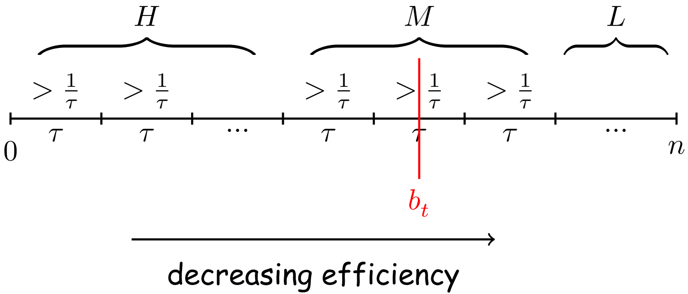
Dealing with Bad Groups



Dealing with Bad Groups

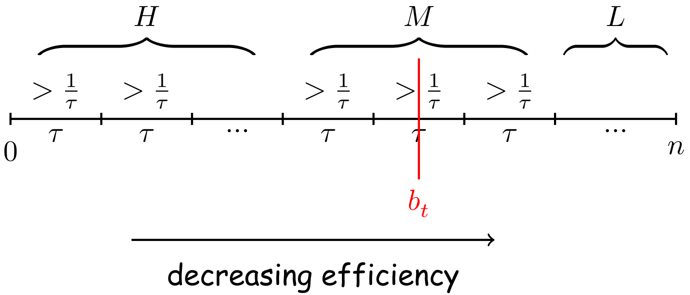


Dealing with Bad Groups



- $w(H^-) \leq 2\tau$
- $w(M^+) \leq 3 \times 2\tau = 6\tau$
- $w(L^+) \leq 2\tau$

Dealing with Bad Groups



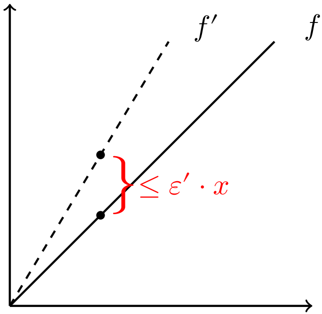
$$f_I(t) = f_H(t_H) + f_M(t_M) + f_L(t_L)$$

- $t_H \geq w(H) - 2\tau$
- $t_M \leq 6\tau$
- $t_L \leq 2\tau$

Dealing with Groups

$f_I(t) = f_H(t_H) + f_M(t_M) + f_L(t_L)$ $\quad$ $\text{err} = O(\frac{1}{\alpha}) \Rightarrow \varepsilon' = \frac{1}{\alpha\tau}$

- f_M, f_L: $t_M \leq 6\tau$, $t_L \leq 2\tau$

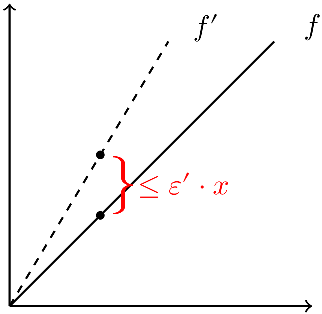


Dealing with Groups

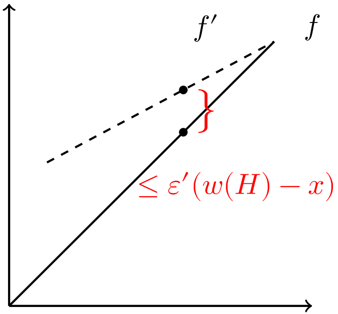
$f_I(t) = f_H(t_H) + f_M(t_M) + f_L(t_L)$

$\text{err} = O(\frac{1}{\alpha}) \Rightarrow \varepsilon' = \frac{1}{\alpha\tau}$

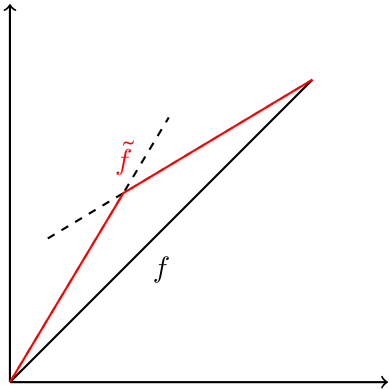
- f_M, f_L: $t_M \leq 6\tau$, $t_L \leq 2\tau$



- f_H: $t_H \geq w(H) - 2\tau$



A Good Approximation for All Capacities



Theorem

There is a $\tilde{O}(n + (\frac{1}{\varepsilon})^{11/6})$-time weak approximation scheme for Knapsack.

Theorem
There is a $\tilde{O}(n + (\frac{1}{\varepsilon})^{11/6})$-time weak approximation scheme for Knapsack.

Can be improved to $\tilde{O}(n + (\frac{1}{\varepsilon})^{7/4})$ by grouping the items in a more careful way.

Future Work

- $\tilde{O}(n + (\frac{1}{\varepsilon})^{3/2})$ time?
  - same as the Bounded Monotone (max,+) Convolution.
- $\tilde{O}(n + (w + p)^{2-\delta})$ time for exact algorithms?
  - under the (min, +)-convolution conjecture, there is no $O(n + w^{2-\delta})$ (or $O(n + p^{2-\delta})$) -time algorithm

Thank you !